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# Some improvements to the extendability of ternary linear codes

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**Abstract**

For a ternary $[n, k, d]$ code $\mathcal{C}$ with $d \equiv 1$ or $2 \pmod 3$, $k \geqslant 3$, the diversity $(\Phi_0, \Phi_1)$ given by

$$\Phi_0 = \frac{1}{2} \sum_{3|i,\, i \neq 0} A_i, \qquad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv 0,\, d \pmod 3} A_i$$

is important to know about the extendability of $\mathcal{C}$, where $A_i$ stands for the number of codewords with weight $i$. As a continuation of [T. Maruta, Extendability of ternary linear codes, Des. Codes Cryptogr. 35 (2005) 175–190], we prove all the conjectures posed in this paper, which yield some improvements to the extendability and non-extendability of ternary linear codes. We also give the necessary and sufficient conditions for the extendability of $\mathcal{C}$ for the case when $k = 4$.


*Keywords:* Extension of linear codes; Ternary linear codes; Diversity; Projective geometry

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## 1. Introduction

Let $V(n,q)$ denote the vector space of $n$-tuples over GF($q$), the field of $q$ elements. We denote the zero vector $(0,0,\ldots,0)$ by $\mathbf{0}$. A $q$-ary linear code $\mathcal{C}$ of length $n$ and dimension $k$ is a $k$-dimensional subspace of $V(n,q)$. The *weight* of a vector $\boldsymbol{x} \in V(n,q)$, denoted by $\mathrm{wt}(\boldsymbol{x})$, is the number of non-zero coordinate positions in $\boldsymbol{x}$. The Hamming distance $d(\boldsymbol{x},\boldsymbol{y})$ between two vectors $\boldsymbol{x},\boldsymbol{y} \in V(n,q)$ is just $\mathrm{wt}(\boldsymbol{x}-\boldsymbol{y})$. The minimum distance of a linear code $\mathcal{C}$ is defined by $d(\mathcal{C}) = \min\{d(\boldsymbol{x},\boldsymbol{y}) \mid \boldsymbol{x},\boldsymbol{y} \in \mathcal{C}, \boldsymbol{x} \neq \boldsymbol{y}\}$. A $q$-ary linear code of length $n$, dimension $k$ and minimum distance $d$ is referred to as an $[n,k,d]_q$ code. The weight distribution of $\mathcal{C}$ is the list of numbers $A_i$ which is the number of codewords of $\mathcal{C}$ with weight $i$. The weight distribution with $(A_0, A_d, \ldots) = (1, \alpha, \ldots)$ is also expressed as $0^1 d^\alpha \cdots$. We only consider *non-degenerate* codes having no coordinate which is identically zero. A $k \times n$ matrix having as rows the vectors of a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$. Two $[n,k,d]_q$ codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if there exists a monomial matrix $M$ with entries in GF($q$) such that $\mathcal{C}'$ coincides with $\mathcal{C}M = \{\boldsymbol{c}M \mid \boldsymbol{c} \in \mathcal{C}\}$.

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1,k,d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* (to $\mathcal{C}'$) and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$. In this paper we are mainly concerned with ternary linear codes with dimension $k \geqslant 3$ (see [6] for the case $k \leqslant 2$). See [2,3,7,9,10] for the extendability of $q$-ary linear codes.

Let $\mathcal{C}$ be an $[n,k,d]_3$ code with $k \geqslant 3$, $\gcd(3,d) = 1$. The *diversity* $(\Phi_0, \Phi_1)$ of $\mathcal{C}$ is given as the pair of integers:

$$\Phi_0 = \frac{1}{2}\sum_{3|i,\, i\neq 0} A_i, \qquad \Phi_1 = \frac{1}{2}\sum_{i\not\equiv 0,\, d \pmod 3} A_i,$$

where the notation $x \mid y$ means that $x$ is a divisor of $y$.

Let $\mathcal{D}_k$ be the set of all possible diversities of $[n,k,d]_3$ codes, which has been already determined in [8] for $k \leqslant 6$:

$$\begin{aligned}
\mathcal{D}_3 &= \{(4,0),(1,6),(4,3),(4,6),(7,3)\};\\
\mathcal{D}_4 &= \{(13,0),(4,18),(13,9),(10,15),(16,12),(13,18),(22,9)\};\\
\mathcal{D}_5 &= \{(40,0),(13,54),(40,27),(31,45),(40,36),(40,45),(49,36),\\
&\qquad (40,54),(67,27)\};\\
\mathcal{D}_6 &= \{(121,0),(40,162),(121,81),(94,135),(121,108),(112,126),\\
&\qquad (130,117),(121,135),(148,108),(121,162),(202,81)\}.
\end{aligned}$$

For $k \geqslant 3$, let $\mathcal{D}_k^*$ and $\mathcal{D}_k^+$ be as follows:

$$\begin{aligned}
\mathcal{D}_k^* &= \{(\theta_{k-2},0),(\theta_{k-3},2\cdot 3^{k-2}),(\theta_{k-2},2\cdot 3^{k-2}),(\theta_{k-2}+3^{k-2},3^{k-2})\},\\
\mathcal{D}_k^+ &= \mathcal{D}_k \setminus \mathcal{D}_k^*,
\end{aligned}$$

where $\theta_j = (3^{j+1} - 1)/2$. It is known that $\mathcal{D}_k^*$ is included in $\mathcal{D}_k$ and that the following theorem holds.

**Theorem 1.1.** [8] *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0, \Phi_1)$, $\gcd(3,d) = 1$, $k \geqslant 3$. Then $\mathcal{C}$ is extendable if one of the following conditions holds*:

(1) $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*$,
(2) $\Phi_0 = \theta_{k-3}$,
(3) $\Phi_1 = 0$,
(4) $\Phi_0 + \Phi_1 < \theta_{k-2} + 3^{k-2}$,
(5) $\Phi_0 + \Phi_1 \geqslant \theta_{k-2} + 2 \cdot 3^{k-2}$,
(6) $2\Phi_0 + \Phi_1 \leqslant 2\theta_{k-2}$.

In this paper we determine $\mathcal{D}_k^+$ for all $k \geqslant 5$.

**Theorem 1.2.**

(1) *When $k$ is odd* ($\geqslant 5$):

$$\begin{aligned}\mathcal{D}_k^+ = \big\{&\big(\theta_{k-2}, 3^{k-2}\big)\big\}\\ &\cup \big\{(\theta_{k-2}, \theta_{k-2} - \theta_{U+1+s}), (\theta_{k-2}, \theta_{k-2} + \theta_{U+1+s} + 1) \mid 0 \leqslant s \leqslant U\big\}\\ &\cup \big\{\big(\theta_{k-2} - 3^{U+1+s}, \theta_{k-2} + \theta_{U+s} + 1\big), \big(\theta_{k-2} + 3^{U+1+s}, \theta_{k-2} - \theta_{U+s}\big) \mid\\ &\qquad 1 \leqslant s \leqslant U+1\big\},\end{aligned}$$

*where $U = (k-5)/2$.*

(2) *When $k$ is even* ($\geqslant 6$):

$$\begin{aligned}\mathcal{D}_k^+ = \big\{&\big(\theta_{k-2}, 3^{k-2}\big)\big\}\\ &\cup \big\{\big(\theta_{k-2} - 3^{T+1+s}, \theta_{k-2} + \theta_{T+s} + 1\big), \big(\theta_{k-2} + 3^{T+1+s}, \theta_{k-2} - \theta_{T+s}\big) \mid\\ &\qquad 0 \leqslant s \leqslant T\big\}\\ &\cup \big\{(\theta_{k-2}, \theta_{k-2} - \theta_{T+s}), (\theta_{k-2}, \theta_{k-2} + \theta_{T+s} + 1) \mid 1 \leqslant s \leqslant T\big\},\end{aligned}$$

*where $T = k/2 - 2$.*

Hence, $|\mathcal{D}_k| = 2k - 1$ for all $k \geqslant 3$, where $|T|$ denotes the number of elements in a set $T$. Our main purpose of this paper is to determine all the possible diversities and the corresponding spectra, which yield some important information on the extendability of $\mathcal{C}$, where 'spectrum' is a notion defined geometrically in Section 2. The condition (5) in Theorem 1.1 can be improved as follows.

**Theorem 1.3.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0, \Phi_1)$, $\gcd(3,d) = 1$, $k \geqslant 3$. Then $\mathcal{C}$ is extendable if $\Phi_0 + \Phi_1 > 2\theta_{k-2} + \theta_{k-4} + 1$ $(= \theta_{k-2} + 3^{k-2} + 2 \cdot 3^{k-3})$.*

We define $\Phi_{\mathrm{e}}$ as follows:

$$\Phi_{\mathrm{e}} = \frac{1}{2} \sum_{d<i\equiv d \pmod 3} A_i .$$

For the non-extendability of ternary linear codes the following is known.

**Theorem 1.4.** [8] *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$, $\gcd(3,d)=1$, $k \geqslant 3$. Then $\mathcal{C}$ is not extendable if*

$$\Phi_{\mathrm{e}} < 3^{k-3}.$$

The condition in Theorem 1.4 can be improved according to the diversities in Theorem 1.2, see Theorems 2.9 and 2.10 in Section 2. More results on the non-extendability are also given in Section 2.

When $k=3$, it is proved in [8] that $\mathcal{D}_3^+ = \{(4,3)\}$ and that every $[n,3,d]_3$ code with diversity $(4,3)$, $\gcd(3,d)=1$, is extendable if and only if $\Phi_{\mathrm{e}} > 0$. But the necessary and sufficient conditions for the extendability of $[n,k,d]_3$ codes with diversity in $\mathcal{D}_k^+$ are not known yet for $k \geqslant 4$.

Now, let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(\Phi_0,\Phi_1) \in \mathcal{D}_4^+ = \{(13,9),(10,15),(16,12)\}$, $d \equiv 1$ or $2 \pmod 3$. Then $\mathcal{C}$ is not extendable if $\Phi_{\mathrm{e}} < 3$ by Theorem 1.4. Otherwise, we need more information. Let $G = [g_0, g_1, g_2, g_3]^{\mathrm{T}}$ be a generator matrix of $\mathcal{C}$, $g_j \in V(n,3)$, where $V(n,3)$ is the $n$-dimensional row vector space over GF(3) and $M^{\mathrm{T}}$ stands for the transpose of a matrix $M$. Then any codeword $c \in \mathcal{C}$ can be written as $v \cdot G = v_0 g_0 + v_1 g_1 + v_2 g_2 + v_3 g_3$ for some $v = (v_0, v_1, v_2, v_3) \in V(4,3)$.

Since there are $2\Phi_{\mathrm{e}}$ vectors $v = (v_0, v_1, v_2, v_3) \in V(4,3)$ such that

$$\mathrm{wt}(v \cdot G) \equiv d \pmod 3, \qquad \mathrm{wt}(v \cdot G) > d \tag{1.1}$$

and since $2 \cdot v$ also satisfies (1.1), one can select such vectors $a_1, \ldots, a_{\Phi_{\mathrm{e}}} \in V(4,3)$ any two of which are linearly independent. Similarly, one can find vectors $b_1, \ldots, b_{\Phi_1} \in V(4,3)$ such that any two of them are linearly independent and that

$$\mathrm{wt}(v \cdot G) \not\equiv 0, d \pmod 3. \tag{1.2}$$

Put $F_{\mathrm{e}} = \{a_1, \ldots, a_{\Phi_{\mathrm{e}}}\}$, $F_1 = \{b_1, \ldots, b_{\Phi_1}\}$. See Section 2 for the geometrical meanings of $F_{\mathrm{e}}$ and $F_1$.

**Theorem 1.5.** *Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(16,12)$, $\gcd(3,d)=1$. Then $\mathcal{C}$ is extendable if and only if one of the following conditions holds*:

(1) *there are three vectors $Q_1$, $Q_2$, $Q_3$ in $F_{\mathrm{e}}$ which are linearly dependent*;
(2) *there are three linearly independent vectors $Q_1$, $Q_2$, $Q_3$ in $F_{\mathrm{e}}$ such that there is no linear combination $v$ ($\neq \mathbf{0}$) of $Q_i$ and $Q_j$ $(1 \leqslant i < j \leqslant 3)$ with $\mathrm{wt}(v \cdot G) \equiv 0 \pmod 3$.*

**Theorem 1.6.** *Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(13,9)$, $\gcd(3,d)=1$. Then $\mathcal{C}$ is extendable if and only if one of the following conditions holds*:

(1) *there are three linearly dependent vectors $P_1$, $P_2$, $P_3$ in $F_1$ and three linearly dependent vectors $Q_1$, $Q_2$, $Q_3$ in $F_e$ such that $\langle P_1, P_2\rangle \cap \langle Q_1, Q_2\rangle \neq \{\mathbf{0}\}$, where $\langle V, W\rangle$ stands for the subspace of $V(4,3)$ spanned by $V$ and $W$*;
(2) *there are three linearly independent vectors $P_1$, $P_2$, $P_3$ in $F_1$ such that each of three $\langle P_i, P_j\rangle$'s contains two vectors of $F_e$ and that there is no linear combination $v$ $(\neq \mathbf{0})$ of $P_i$ and $P_j$ with $\mathrm{wt}(v \cdot G) \equiv 0 \pmod 3$ for $(1 \leqslant i < j \leqslant 3)$.*

**Theorem 1.7.** *Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(10,15)$, $\gcd(3,d)=1$. Then $\mathcal{C}$ is extendable if and only if one of the following conditions holds*:

(1) *there are three linearly independent vectors $Q_1$, $Q_2$, $Q_3$ in $F_e$ such that there is no linear combination $v$ $(\neq \mathbf{0})$ of $Q_i$ and $Q_j$ $(1 \leqslant i < j \leqslant 3)$ with $\mathrm{wt}(v \cdot G) \equiv 0 \pmod 3$*;
(2) *there are three linearly dependent vectors $Q_1$, $Q_2$, $Q_3$ and three linearly dependent vectors $Q_1'$, $Q_2'$, $Q_3'$ in $F_e$ such that $\langle Q_1, Q_2\rangle \neq \langle Q_1', Q_2'\rangle$, $\langle Q_1, Q_2\rangle \cap \langle Q_1', Q_2'\rangle \neq \{\mathbf{0}\}$ and that $v \cdot G \equiv 0 \pmod 3$ holds for every $v \in \langle Q_1, Q_2\rangle \cap \langle Q_1', Q_2'\rangle$*;
(3) *there are three linearly independent vectors $P_1$, $P_2$, $P_3$ in $F_1$ such that each of three $\langle P_i, P_j\rangle$'s contains two vectors of $F_e$ and that there is no linear combination $v$ $(\neq \mathbf{0})$ of $P_i$ and $P_j$ with $\mathrm{wt}(v \cdot G) \equiv 0 \pmod 3$ for $(1 \leqslant i < j \leqslant 3)$.*

Note that the conditions (2) of Theorem 1.6 and (2), (3) of Theorem 1.7 require that $\Phi_e \geqslant 6$.

**Theorem 1.8.** *Let $\mathcal{C}$ be an extendable $[n,4,d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_4^+$, $\gcd(3,d)=1$. Then the vector $h$ $(\neq \mathbf{0}) \in V(4,3)$ such that $\mathcal{C}$ is extended by adding $h^{\mathrm{T}}$ to $G$ can be taken as follows*:

(1) *$h$ is orthogonal to $\langle Q_1, Q_2, Q_3\rangle$ when $(\Phi_0, \Phi_1) = (16,12)$ and (2) of Theorem 1.5 is satisfied or when $(\Phi_0, \Phi_1) = (10,15)$ and (1) of Theorem 1.7 is satisfied*;
(2) *$h$ is orthogonal to $\langle P_1, P_2, P_3\rangle$ when $(\Phi_0, \Phi_1) = (13,9)$ and (2) of Theorem 1.6 is satisfied or when $(\Phi_0, \Phi_1) = (10,15)$ and (3) of Theorem 1.7 is satisfied*;
(3) *$h$ is orthogonal to $\langle Q_1, Q_2, Q_1'\rangle$ when $(\Phi_0, \Phi_1) = (10,15)$ and (2) of Theorem 1.7 is satisfied.*

We prove Theorems 1.5–1.8 in Section 3.

**Example 1.1.** Let $\mathcal{C}_0$ be a $[22,4,13]_3$ code with a generator matrix

$$G_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&2&2&2 \\ 0&0&1&0&1&1&0&0&1&1&0&0&1&1&0&1&1&2&2&0&1&2 \\ 0&0&0&1&1&2&1&2&0&1&1&2&0&1&0&0&1&0&2&1&1&0 \end{bmatrix}$$

whose weight distribution is $0^1 13^{20} 14^{12} 15^{28} 16^4 17^{12} 18^4$ (diversity (16,12), $\Phi_{\mathrm{e}} = 2$). Then $\mathcal{C}_0$ is not extendable by Theorem 1.4.

**Example 1.2.** Let $G(z_1, z_2, z_3, z_4)$ be the $4 \times 13$ matrix defined by

$$G(z_1, z_2, z_3, z_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & z_1 & z_2 & z_3 & z_4 \end{bmatrix}.$$

(1) Let $\mathcal{C}_1$ be a $[13, 4, 7]_3$ code with a generator matrix $G_1 = G(1, 0, 2, 2)$, whose weight distribution is $0^1 7^{18} 8^{16} 9^{28} 10^6 11^8 12^4$ (diversity (16, 12), $\Phi_{\mathrm{e}} = 3$). Then we can take $F_{\mathrm{e}}$ as $F_{\mathrm{e}} = \{a_1 = (1, 2, 2, 0), a_2 = (1, 2, 0, 1), a_3 = (0, 0, 1, 1)\}$. Since $a_1$, $a_2$, $a_3$ are linearly dependent, $\mathcal{C}_1$ is extendable by Theorem 1.5. Indeed, we can take $h^{\mathrm{T}} = (1, 2, 2, 1)$ so that $[G_1, h]$ generates a $[14, 4, 8]_3$ code with weight distribution $0^1 8^{24} 9^{24} 10^{20} 12^8 13^4$.

(2) Let $\mathcal{C}_2$ be a $[13, 4, 7]_3$ code with a generator matrix $G_2 = G(0, 2, 1, 1)$, whose weight distribution is $0^1 7^{16} 8^{18} 9^{30} 10^6 11^6 12^2 13^2$ (diversity (16, 12), $\Phi_{\mathrm{e}} = 4$). Then we can take $F_{\mathrm{e}}$ as $F_{\mathrm{e}} = \{a_1 = (1, 2, 2, 0), a_2 = (1, 2, 0, 2), a_3 = (1, 0, 2, 2), a_4 = (1, 1, 1, 1)\}$. Since $a_1$, $a_2$, $a_3$ satisfies the condition (2) of Theorem 1.5, $\mathcal{C}_2$ is extendable. By Theorem 1.8, we can take $h^{\mathrm{T}} = (1, 2, 2, 2)$ so that $[G_2, h]$ generates an extension of $\mathcal{C}_2$ whose weight distribution is $0^1 8^{28} 9^{12} 10^{30} 12^8 14^2$.

(3) Let $\mathcal{C}_3$ be a $[13, 4, 7]_3$ code with a generator matrix $G_3 = G(1, 2, 1, 0)$, whose weight distribution is $0^1 7^{18} 8^{16} 9^{28} 10^6 11^8 12^4$ (just the same as $\mathcal{C}_1$'s!). Then we can take $F_{\mathrm{e}}$ as $F_{\mathrm{e}} = \{a_1 = (1, 2, 2, 0), a_2 = (1, 2, 2, 1), a_3 = (0, 0, 1, 2)\}$. Since $a_1$, $a_2$, $a_3$ are linearly independent and since $v = a_1 + a_2 = (2, 1, 1, 1)$ satisfies $\mathrm{wt}(v \cdot G) \equiv 0 \pmod 3$, $\mathcal{C}_3$ is not extendable by Theorem 1.5.

Other examples where one can apply Theorems 1.6 and 1.7 are given in Section 3.

## 2. Diversities of ternary linear codes

We first introduce preliminary notions and results from [8] which are needed for proving our new results presented in Section 1.

The projective geometry of dimension $r$ over $\mathrm{GF}(q)$ is denoted by $\mathrm{PG}(r, q)$. A *$j$-flat* is a projective subspace of dimension $j$ in $\mathrm{PG}(r, q)$. 0-Flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively [4]. We denote by $\mathcal{F}_j$ the set of $j$-flats of $\mathrm{PG}(r, q)$ and denote by $\theta_j$ the number of points in a $j$-flat, i.e., $\theta_j = |\mathrm{PG}(j, q)| = (q^{j+1} - 1)/(q - 1)$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix $G$. Then the columns of $G$ can be considered as a multiset of $n$ points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted also by $\mathcal{C}$. An *$i$-point* is a point of $\Sigma$ which has multiplicity $i$ in $\mathcal{C}$. Denote by $\gamma_0$ the maximum multiplicity of a point from $\Sigma$ in $\mathcal{C}$ and let $C_i$ be the set of $i$-points in $\Sigma$, $0 \leqslant i \leqslant \gamma_0$. For any subset $S$ of $\Sigma$ we define *the multiplicity of $S$ with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|.$$

Note that the number of hyperplanes $\pi$ with $m_{\mathcal{C}}(\pi) = i$ is equal to $A_{n-i}/(q-1)$ for $0 \leqslant i \leqslant n-d$. Then we obtain the partition $\Sigma = C_0 \cup C_1 \cup \cdots \cup C_{\gamma_0}$ such that

$$n = m_{\mathcal{C}}(\Sigma), \qquad n - d = \max\bigl\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\bigr\}.$$

Conversely such a partition of $\Sigma$ as above gives an $[n,k,d]_q$ code in the natural manner if there exists no hyperplane containing the complement of $C_0$ in $\Sigma$.

Since (*length*) − (*minimum distance*) $= n - d$ also holds for an extension of $\mathcal{C}$, we get the following.

**Lemma 2.1.** *$\mathcal{C}$ is extendable if and only if there exists a point $P \in \Sigma$ such that $m_{\mathcal{C}}(\pi) < n - d$ for all hyperplanes $\pi$ through $P$.*

Let $\Sigma^*$ be the dual space of $\Sigma$ (considering $\mathcal{F}_{k-2}$ as the set of points of $\Sigma^*$). Then Lemma 2.1 is equivalent to the following:

**Lemma 2.2.** *$\mathcal{C}$ is extendable if and only if there exists a hyperplane $\Pi$ of $\Sigma^*$ such that*

$$\Pi \subset \bigl\{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) < n - d\bigr\}.$$

From now on, we assume that $\mathcal{C}$ is an $[n,k,d]_3$ code with diversity $(\Phi_0, \Phi_1)$, $\gcd(3,d) = 1$, $k \geqslant 3$. Define $F_0$, $F_1$, $F_{\mathrm{e}}$, $F$ and $\bar{F}$ as follows:

$$\begin{aligned}
F_0 &= \bigl\{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) \equiv n \pmod 3\bigr\},\\
F_1 &= \bigl\{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) \not\equiv n,\ n-d \pmod 3\bigr\},\\
F_{\mathrm{e}} &= \bigl\{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) < n-d,\ m_{\mathcal{C}}(\pi) \equiv n-d \pmod 3\bigr\},\\
F &= F_0 \cup F_1, \qquad \bar{F} = F \cup F_{\mathrm{e}}.
\end{aligned}$$

Then $F$ forms a blocking set with respect to lines in the dual space $\Sigma^*$ of $\Sigma = \mathrm{PG}\,(k-1,3)$ [7], that is, every line of $\Sigma^*$ meets $F$ in at least one point of $\Sigma^*$. The following is straightforward from Lemma 2.2.

**Lemma 2.3.** *$\mathcal{C}$ is extendable if and only if $\bar{F}$ contains a hyperplane of $\Sigma^*$.*

Note that $|F_0| = \Phi_0$, $|F_1| = \Phi_1$, $|F_{\mathrm{e}}| = \Phi_{\mathrm{e}}$. A subset $S^*$ of $\Sigma^*$ with $|S^* \cap F_0| = i$, $|S^* \cap F_1| = j$ is called an *$(i,j)$-set*. A line of $\Sigma^*$ which forms an $(i,j)$-set is called an

$(i, j)$-*line*. An $(i, j)$-*plane*, an $(i, j)$-*solid*, an $(i, j)$-*secundum* and an $(i, j)$-*hyperplane* are defined similarly.

We denote by $\mathcal{F}_j^*$ the set of $j$-flats of $\Sigma^*$, so $\mathcal{F}_j^* = \mathcal{F}_{k-2-j}$, $0 \leqslant j \leqslant k-2$. Let $\Lambda_1$ be the set of all possible $(i, j)$ for which an $(i, j)$-line exists in $\mathcal{F}_1^*$. Then we have

$$\Lambda_1 = \{(1,0), (0,2), (2,1), (1,3), (4,0)\},$$

see [8]. Assume $2 \leqslant t \leqslant k-1$ and let $\delta_t \in \mathcal{F}_t^*$. Denote by $c_t(i, j)$ $(= c_{i,j}^{(t)}$ in [8]) the number of $(t-1)$-flats which form $(i, j)$-sets in $\delta_t$ and let ${\varphi_s}^{(t)} = |\delta_t \cap F_s|$, $s = 0, 1$. $(\varphi_0^{(t)}, \varphi_1^{(t)})$ is called the *diversity of* $\delta_t$ and the list of $c_t(i, j)$'s is called its *spectrum*. An easy counting argument yields the following:

$$\sum_{(i,j)\in\Lambda_{t-1}} c_t(i, j) = \theta_t, \qquad \sum_{(i,j)\in\Lambda_{t-1}} i \cdot c_t(i, j) = \theta_{t-1}{\varphi_0}^{(t)},$$

$$\sum_{(i,j)\in\Lambda_{t-1}} j \cdot c_t(i, j) = \theta_{t-1}{\varphi_1}^{(t)}, \qquad \sum_{(i,j)\in\Lambda_{t-1}} \binom{i}{2} c_t(i, j) = \theta_{t-2}\binom{{\varphi_0}^{(t)}}{2},$$

$$\sum_{(i,j)\in\Lambda_{t-1}} \binom{j}{2} c_t(i, j) = \theta_{t-2}\binom{{\varphi_1}^{(t)}}{2},$$

$$\sum_{(i,j)\in\Lambda_{t-1}} \binom{i+j}{2} c_t(i, j) = \theta_{t-2}\binom{{\varphi_0}^{(t)} + {\varphi_1}^{(t)}}{2},$$

where $\Lambda_{t-1}$ is the set of all possible $(\varphi_0^{(t-1)}, \varphi_1^{(t-1)})$. We refer to the above simultaneous six equations as $(*)$. Note that the last equation of $(*)$ can be replaced by

$$\sum_{(i,j)\in\Lambda_{t-1}} i \cdot j \cdot c_t(i, j) = \theta_{t-2} \cdot \varphi_0^{(t)} \varphi_1^{(t)}.$$

Assume $t = 2$. The above six equations $(*)$ have the seven solutions in Table 2.1 [8].

We use the following geometric characterization to find the necessary and sufficient conditions for the extendability of ternary linear codes of dimension 4 in Section 3.

Table 2.1

| Type | $\varphi_0^{(2)}$ | $\varphi_1^{(2)}$ | $c_2(1,0)$ | $c_2(0,2)$ | $c_2(2,1)$ | $c_2(1,3)$ | $c_2(4,0)$ |
|---|---|---|---|---|---|---|---|
| (a-1) | 4 | 0 | 12 | 0 | 0 | 0 | 1 |
| (a-2) | 1 | 6 | 2 | 9 | 0 | 2 | 0 |
| (a-3) | 4 | 3 | 4 | 3 | 6 | 0 | 0 |
| (a-4) | 4 | 6 | 0 | 3 | 6 | 4 | 0 |
| (a-5) | 7 | 3 | 1 | 0 | 9 | 1 | 2 |
| (a-6) | 4 | 9 | 0 | 0 | 0 | 12 | 1 |
| (a-7) | 13 | 0 | 0 | 0 | 0 | 0 | 13 |

Table 2.2

| $\varphi_0^{(3)}$ | $\varphi_1^{(3)}$ | $c_3(4,0)$ | $c_3(1,6)$ | $c_3(4,3)$ | $c_3(4,6)$ | $c_3(7,3)$ | $c_3(4,9)$ | $c_3(13,0)$ |
|---|---|---|---|---|---|---|---|---|
| 13 | 0 | 39 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 18 | 2 | 36 | 0 | 0 | 0 | 2 | 0 |
| 13 | 9 | 4 | 3 | 27 | 0 | 6 | 0 | 0 |
| 10 | 15 | 0 | 10 | 15 | 15 | 0 | 0 | 0 |
| 16 | 12 | 0 | 0 | 12 | 12 | 16 | 0 | 0 |
| 13 | 18 | 0 | 3 | 0 | 27 | 6 | 4 | 0 |
| 22 | 9 | 1 | 0 | 0 | 0 | 36 | 1 | 2 |
| 13 | 27 | 0 | 0 | 0 | 0 | 0 | 39 | 1 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 40 |

**Lemma 2.4.** [8] *The geometrical structure of $\delta_2 \cap F$ for each case in Table* 2.1 *is the following*:

(a-1) *a line contained in $F_0$*;
(a-2) *two $(1,3)$-lines meeting in the point of $F_0$*;
(a-3) $\delta_2 \cap F_0 = K$ *and* $\delta_2 \cap F_1 = \{Q_1, Q_2, Q_3\}$;
(a-4) $\delta_2 \cap F_0 = K$ *and* $\delta_2 \cap F_1 = \delta_2 \setminus (K \cup \{Q_1, Q_2, Q_3\})$;
(a-5) *two $(4,0)$-lines $l_1$, $l_2$ and a $(1,3)$-line through $l_1 \cap l_2$*;
(a-6) *a line $l$ contained in $F_0$ and $\delta_2 \setminus l \subset F_1$*;
(a-7) $\delta_2 \subset F_0$,

*where $K = \{P_1, P_2, P_3, P_4\}$ is a $(4,2)$-arc* (*that is, a set of four points in* PG$(2,3)$ *no three of which are collinear*) *and $Q_1 = \langle P_1, P_2\rangle \cap \langle P_3, P_4\rangle$, $Q_2 = \langle P_1, P_3\rangle \cap \langle P_2, P_4\rangle$, $Q_3 = \langle P_1, P_4\rangle \cap \langle P_2, P_3\rangle$.* (*$\langle P_i, P_j\rangle$ stands for the line through $P_i$ and $P_j$. See* [4] *for arcs.*)

For $t = 3$, six equations $(*)$ have the nine solutions in Table 2.2 [8].
From Tables 2.1 and 2.2, we have

$$\Lambda_2 = \bigl\{(4,0), (1,6), (4,3), (4,6), (7,3), (4,9), (13,0)\bigr\},$$
$$\Lambda_3 = \bigl\{(13,0), (4,18), (13,9), (10,15), (16,12), (13,18), (22,9), (13,27), (40,0)\bigr\}.$$

For $t \geqslant 3$, let $\Lambda_t^+$ be the set of all possible $(\varphi_0^{(t)}, \varphi_1^{(t)}) \in \Lambda_t$ such that $\delta_t$ contains a $(4,3)$-plane and let $\Lambda_t^-$ be the set of all possible $(\varphi_0^{(t)}, \varphi_1^{(t)}) \in \Lambda_t$ such that $\delta_t$ contains no $(4,3)$-plane.

**Lemma 2.5.** [8] *For $t \geqslant 3$, $\Lambda_t^-$ is given as*

$$\Lambda_t^- = \bigl\{(\theta_{t-1}, 0), \bigl(\theta_{t-2}, 2\cdot 3^{t-1}\bigr), \bigl(\theta_{t-1}, 2\cdot 3^{t-1}\bigr), \bigl(\theta_{t-1} + 3^{t-1}, 3^{t-1}\bigr), \bigl(\theta_{t-1}, 3^t\bigr), (\theta_t, 0)\bigr\}$$

*and the spectrum corresponding to each diversity is uniquely determined as follows*:

$$\big(c_t(\theta_{t-2},0),c_t(\theta_{t-1},0)\big)=(\theta_t-1,1)\quad \textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=(\theta_{t-1},0);$$
$$\big(c_t(\theta_{t-2},0),c_t\big(\theta_{t-3},2\cdot 3^{t-2}\big),c_t\big(\theta_{t-2},3^{t-1}\big)\big)=(2,\theta_t-\theta_1,2)$$
$$\textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=\big(\theta_{t-2},2\cdot 3^{t-1}\big);$$
$$\big(c_t\big(\theta_{t-3},2\cdot 3^{t-2}\big),c_t\big(\theta_{t-2},2\cdot 3^{t-2}\big),c_t\big(\theta_{t-2}+3^{t-2},3^{t-2}\big),c_t\big(\theta_{t-2},3^{t-1}\big)\big)$$
$$=(3,\theta_t-\theta_2,6,4)\quad \textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=\big(\theta_{t-1},2\cdot 3^{t-1}\big);$$
$$\big(c_t(\theta_{t-2},0),c_t\big(\theta_{t-2}+3^{t-2},3^{t-2}\big),c_t\big(\theta_{t-2},3^{t-1}\big),c_t(\theta_{t-1},0)\big)=(1,\theta_t-\theta_1,1,2)$$
$$\textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=\big(\theta_{t-1}+3^{t-1},3^{t-1}\big);$$
$$\big(c_t\big(\theta_{t-2},3^{t-1}\big),c_t(\theta_{t-1},0)\big)=(\theta_t-1,1)\quad \textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=\big(\theta_{t-1},3^{t}\big);$$
$$c_t(\theta_{t-1},0)=\theta_t\quad \textit{for}\ \big(\varphi_0^{(t)},\varphi_1^{(t)}\big)=(\theta_t,0).$$

**Lemma 2.6.** [8] $\Lambda_t^-\cap\Lambda_t^+=\emptyset$.

It follows from Lemmas 2.5 and 2.6 that

$$\mathcal{D}_k^*=\Lambda_{k-1}^-\setminus\big\{\big(\theta_{k-2},3^{k-1}\big),(\theta_{k-1},0)\big\},\qquad \mathcal{D}_k^+=\Lambda_{k-1}^+.$$

**Lemma 2.7.** [8] *For* $(\varphi_0^{(t)},\varphi_1^{(t)})\in\Lambda_t^+$, $t\geqslant 3$, $c_t(\theta_{t-2},0)>0$ *implies*

$$\begin{aligned}&\big(\varphi_0^{(t)},\varphi_1^{(t)};c_t(\theta_{t-2},0),c_t\big(\theta_{t-3},2\cdot 3^{t-2}\big),c_t\big(\theta_{t-2},3^{t-2}\big),c_t\big(\theta_{t-2}+3^{t-2},3^{t-2}\big)\big)\\&\quad=\big(\theta_{t-1},3^{t-1};4,3,\theta_t-\theta_2,6\big).\end{aligned}$$

A diversity $(\varphi_0^{(t)},\varphi_1^{(t)})\in\Lambda_t$ is called *new* if $((\varphi_0^{(t)}-1)/3,\varphi_1^{(t)}/3)\notin\Lambda_{t-1}$. Diversities of types (A-1), (A-2), (B-1), (B-2) in the next theorem are new in $\Lambda_t$, which were found by Tsunakawa [11].

**Theorem 2.8.**

(1) *When $t$ is odd* ($\geqslant 5$):

$$\begin{aligned}\Lambda_t^+=&\big\{\big(\theta_{t-1},3^{t-1}\big)\big\}\\&\cup\big\{\big(\theta_{t-1}-3^{T+1+s},\theta_{t-1}+\theta_{T+s}+1\big),\big(\theta_{t-1}+3^{T+1+s},\theta_{t-1}-\theta_{T+s}\big)\ \big|\ 0\leqslant s\leqslant T\big\}\\&\cup\big\{(\theta_{t-1},\theta_{t-1}-\theta_{T+s}),(\theta_{t-1},\theta_{t-1}+\theta_{T+s}+1)\mid 1\leqslant s\leqslant T\big\},\end{aligned}$$

*where $T=(t-3)/2$. The spectrum corresponding to each diversity is uniquely determined as follows*:

(A-1) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$;
$c_t(\theta_{t-2} - 3^{T+1}, \theta_{t-2} + \theta_T + 1) = \theta_{t-1} - 3^{T+1}$,
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_T) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_T + 1) = \theta_{t-1} + \theta_T + 1$,
(A-2) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_T) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_T + 1) = \theta_{t-1} - \theta_T$,
$c_t(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T) = \theta_{t-1} + 3^{T+1}$,
(A-3) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^{t-1})$;
$(c_t(\theta_{t-2}, 0), c_t(\theta_{t-3}, 2 \cdot 3^{t-2}), c_t(\theta_{t-2}, 3^{t-2})$,
$c_t(\theta_{t-2} + 3^{t-2}, 3^{t-2})) = (4, 3, \theta_t - \theta_2, 6)$,
(A-4) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1+s}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \leqslant s \leqslant T$;
$c_t(\theta_{t-2} - 3^{T+1+s}, \theta_{t-2} + \theta_{T+s} + 1) = \theta_{t-1-2s} - 3^{T+1-s}$,
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{T+s}) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_{T+s} + 1) = \theta_{t-1-2s} + \theta_{T-s} + 1$,
$c_t(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1) = \theta_t - \theta_{t-2s}$,
(A-5) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{T+1+s}, \theta_{t-1} - \theta_{T+s})$, $1 \leqslant s \leqslant T$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{T+s}) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_{T+s} + 1) = \theta_{t-1-2s} - \theta_{T-s}$,
$c_t(\theta_{t-2} + 3^{T+1+s}, \theta_{t-2} - \theta_{T+s}) = \theta_{t-1-2s} + 3^{T+1-s}$,
$c_t(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s}) = \theta_t - \theta_{t-2s}$,
(A-6) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} - \theta_{T+s})$, $1 \leqslant s \leqslant T$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{T+s}) = \theta_{t-2s}$,
$c_t(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1) = \theta_{t-2s} - \theta_{T+1-s}$,
$c_t(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s}) = \theta_{t-2s} + \theta_{T+1-s} + 1$,
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{T-1+s}) = \theta_t - \theta_{t+1-2s}$,
(A-7) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \leqslant s \leqslant T$;
$c_t(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1) = \theta_{t-2s} - \theta_{T+1-s}$,
$c_t(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s}) = \theta_{t-2s} + \theta_{T+1-s} + 1$,
$c_t(\theta_{t-2}, \theta_{t-2} + \theta_{T+s} + 1) = \theta_{t-2s}$, $c_t(\theta_{t-2}, \theta_{t-2} + \theta_{T-1+s} + 1) = \theta_t - \theta_{t+1-2s}$.

(2) *When $t$ is even* ($\geqslant 4$):

$$\begin{aligned}\Lambda_t^+ = &\big\{\big(\theta_{t-1}, 3^{t-1}\big)\big\}\\ &\cup \big\{(\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1) \mid 0 \leqslant s \leqslant U\big\}\\ &\cup \big\{\big(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1\big), \big(\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s}\big) \bigm| \\ &\quad 1 \leqslant s \leqslant U + 1\big\},\end{aligned}$$

*where $U = (t-4)/2$. The spectrum corresponding to each diversity is uniquely determined as follows*:

(B-1) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1})$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{U+1}) = \theta_{t-1}$, $c_t(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1) = \theta_{t-1} - \theta_{U+1}$,
$c_t(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U) = \theta_{t-1} + \theta_{U+1} + 1$,
(B-2) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)$;

$c_t(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1) = \theta_{t-1} - \theta_{U+1}$, $c_t(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U) = \theta_{t-1} + \theta_{U+1} + 1$,
$c_t(\theta_{t-2}, \theta_{t-2} + \theta_{U+1} + 1) = \theta_{t-1}$,

(B-3) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^{t-1})$;
$(c_t(\theta_{t-2}, 0), c_t(\theta_{t-3}, 2 \cdot 3^{t-2}), c_t(\theta_{t-2}, 3^{t-2}), c_t(\theta_{t-2} + 3^{t-2}, 3^{t-2})) = (4, 3, \theta_t - \theta_2, 6)$,

(B-4) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1)$, $1 \leqslant s \leqslant U + 1$;
$c_t(\theta_{t-2} - 3^{U+1+s}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_{t-2s} - 3^{U+2-s}$,
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{U+s}) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_{t-2s} + \theta_{U+1-s} + 1$,
$c_t(\theta_{t-2} - 3^{U+s}, \theta_{t-2} + \theta_{U-1+s} + 1) = \theta_t - \theta_{t+1-2s}$,

(B-5) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s})$, $1 \leqslant s \leqslant U + 1$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{U+s}) = c_t(\theta_{t-2}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_{t-2s} - \theta_{U+1-s}$,
$c_t(\theta_{t-2} + 3^{U+1+s}, \theta_{t-2} - \theta_{U+s}) = \theta_{t-2s} + 3^{U+2-s}$,
$c_t(\theta_{t-2} + 3^{U+s}, \theta_{t-2} - \theta_{U-1+s}) = \theta_t - \theta_{t+1-2s}$,

(B-6) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s})$, $1 \leqslant s \leqslant U$;
$c_t(\theta_{t-2}, \theta_{t-2} - \theta_{U+1+s}) = \theta_{t-1-2s}$, $c_t(\theta_{t-2} - 3^{U+1+s}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_{t-1-2s} - \theta_{U+1-s}$,
$c_t(\theta_{t-2} + 3^{U+1+s}, \theta_{t-2} - \theta_{U+s}) = \theta_{t-1-2s} + \theta_{U+1-s} + 1$, $c_t(\theta_{t-2}, \theta_{t-2} - \theta_{U+s}) = \theta_t - \theta_{t-2s}$,

(B-7) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1)$, $1 \leqslant s \leqslant U$;
$c_t(\theta_{t-2} - 3^{U+1+s}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_{t-1-2s} - \theta_{U+1-s}$,
$c_t(\theta_{t-2} + 3^{U+1+s}, \theta_{t-2} - \theta_{U+s}) = \theta_{t-1-2s} + \theta_{U+1-s} + 1$,
$c_t(\theta_{t-2}, \theta_{t-2} + \theta_{U+1+s} + 1) = \theta_{t-1-2s}$, $c_t(\theta_{t-2}, \theta_{t-2} + \theta_{U+s} + 1) = \theta_t - \theta_{t-2s}$.

**Proof.** It is easily checked that (A-1)–(A-7) and (B-1)–(B-7) are solutions of the simultaneous six equations $(*)$ when $t$ is odd ($\geqslant 5$) and when $t$ is even ($\geqslant 4$), respectively. We prove that these are the only solutions by induction on $t$. Since this is already proved in [8] for $t = 4, 5$, we assume that $\Lambda_{t_0}^+$ and the corresponding spectra are given as in the theorem for all $t_0$, $4 \leqslant t_0 \leqslant t$, $t \geqslant 5$. We solve $(*)$ for $\delta_{t+1}$ containing a $(4, 3)$-plane. Let $M$ be the coefficient matrix of $(*)$, that is a $6 \times |\Lambda_t|$ matrix consisting of the columns $M_{i,j}$:

$$M_{i,j} = \left[1, i, j, \binom{i}{2}, \binom{j}{2}, \binom{i+j}{2}\right]^{\mathrm{T}}, \quad (i, j) \in \Lambda_t.$$

Assume $t$ is odd ($\geqslant 5$).

First we assume that $c_{t+1}(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1) > 0$ and let $\Pi$ be a $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$-hyperplane of $\delta_{t+1}$. Then every hyperplane ($\neq \Pi$) of $\delta_{t+1}$ meets $\Pi$ in an $(i, j)$-secundum of $\delta_{t+1}$ with $(i, j) \in \{(\theta_{t-2} - 3^{T+1}, \theta_{t-2} + \theta_T + 1), (\theta_{t-2}, \theta_{t-2} - \theta_T), (\theta_{t-2}, \theta_{t-2} + \theta_T + 1)\}$. Hence $c_{t+1}(i, j) > 0$ implies $(i, j) \in \Lambda'_t := \{(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1), (\theta_{t-1}, \theta_{t-1} - \theta_{T+1}), (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1), (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T), (\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1)\}$. Let $M'$ be the $5 \times 5$ submatrix of $M$ restricting to the five columns which correspond to these five diversities in $\Lambda'_t$ and the first five rows of $M$. Then it can be calculated that the rank of $M'$ is five, so is the rank of $M$. Hence the six equations $(*)$ have at most one solution for given $(\varphi_0^{(t+1)}, \varphi_1^{(t+1)})$.

- If $c_{t+1}(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1) > 0$, then all the hyperplanes through a fixed $(\theta_{t-2} - 3^{T+2}, \theta_{t-2} + \theta_{T+1} + 1)$-secundum of $\delta_{t+1}$ have diversity of type (A-4) with $s = 1$, so

$$\varphi_0^{(t+1)} = \left(\theta_{t-1} - 3^{T+2} - \theta_{t-2} + 3^{T+2}\right)\theta_1 + \theta_{t-2} - 3^{T+2} = \theta_t - 3^{T+2},$$
$$\varphi_1^{(t+1)} = (\theta_{t-1} + \theta_{T+1} + 1 - \theta_{t-2} - \theta_{T+1} - 1)\theta_1 + \theta_{t-2} + \theta_{T+1} + 1 = \theta_t + \theta_{T+1} + 1,$$

giving the diversity of type (B-4) with $s = 1$ for $\delta_{t+1}$.
- If $c_{t+1}(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1) = 0$ and $c_{t+1}(\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1) > 0$, then we get

$$\varphi_0^{(t+1)} = (\theta_{t-1} - \theta_{t-2})\theta_1 + \theta_{t-2} = \theta_t,$$
$$\varphi_1^{(t+1)} = (\theta_{t-1} + \theta_{T+1} + 1 - \theta_{t-2} - \theta_{T+1} - 1)\theta_1 + \theta_{t-2} + \theta_{T+1} + 1 = \theta_t + \theta_{T+1} + 1,$$

giving the diversity of type (B-2) for $\delta_{t+1}$.
- If $c_{t+1}(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1) = c_{t+1}(\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1) = 0$ and $c_{t+1}(\theta_{t-1}, \theta_{t-1} - \theta_{T+1}) > 0$, then we obtain

$$\varphi_0^{(t+1)} = (\theta_{t-1} - \theta_{t-2})\theta_1 + \theta_{t-2} = \theta_t,$$
$$\varphi_1^{(t+1)} = (\theta_{t-1} - \theta_{T+1} - \theta_{t-2} + \theta_{T+1})\theta_1 + \theta_{t-2} - \theta_{T+1} = \theta_t - \theta_{T+1},$$

giving the diversity of type (B-1) for $\delta_{t+1}$.
- If $c_{t+1}(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1) = c_{t+1}(\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1) = c_{t+1}(\theta_{t-1}, \theta_{t-1} - \theta_{T+1}) = 0$, then considering the hyperplanes through a fixed $(\theta_{t-2} - 3^{T+1}, \theta_{t-2} + \theta_T + 1)$-secundum in $\delta_{t+1}$, we get $\varphi_0^{(t+1)} = (\theta_{t-1} - 3^{T+1} - \theta_{t-2} + 3^{T+1})\theta_1 + \theta_{t-2} - 3^{T+1} = \theta_t - 3^{T+1}$, so $c_{t+1}(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T) = 0$ (otherwise, $\varphi_0^{(t+1)} = (\theta_{t-1} + 3^{T+1} - \theta_{t-2} - 3^{T+1})\theta_1 + \theta_{t-2} + 3^{T+1} = \theta_t + 3^{T+1}$, a contradiction). Considering the hyperplanes through a fixed $(\theta_{t-2}, \theta_{t-2} - \theta_T)$-secundum in $\delta_{t+1}$, we have $\varphi_0^{(t+1)} = (\theta_{t-1} - 3^{T+1} - \theta_{t-2})\theta_1 + \theta_{t-2} = \theta_t - 4 \cdot 3^{T+1}$, a contradiction. Hence there is no solution of $(*)$ when $c_{t+1}(\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1) = c_{t+1}(\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1) = c_{t+1}(\theta_{t-1}, \theta_{t-1} - \theta_{T+1}) = 0$.

Next we assume that $c_{t+1}(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T) > 0$. Considering similarly as above, $c_{t+1}(i, j) > 0$ implies $(i, j) \in \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+1}), (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1), (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T), (\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1), (\theta_{t-1} + 3^{T+2}, \theta_{t-1} - \theta_{T+1})\}$, and it also holds that rank $M = 5$. Hence the six equations $(*)$ have at most one solution for given $(\varphi_0^{(t+1)}, \varphi_1^{(t+1)})$. We may assume that $c_{t+1}(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1) = 0$. If $c_{t+1}(\theta_{t-1} + 3^{T+2}, \theta_{t-1} - \theta_{T+1}) > 0$, then we get the diversity of type (B-5) with $s = 1$ for $\delta_{t+1}$. Otherwise the six equations $(*)$ have no solution.

Now, we assume that there is no hyperplane of $\delta_{t+1}$ whose diversity is new in $\Lambda_t$, i.e., $c_{t+1}(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1) = c_{t+1}(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T) = 0$.

Assume that $c_{t+1}(\theta_{t-1}, \theta_{t-1} - \theta_{T+1}) > 0$. Then $c_{t+1}(i, j) > 0$ implies $(i, j) \in \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+2}), (\theta_{t-1} - 3^{T+2}, \theta_{t-1} + \theta_{T+1} + 1), (\theta_{t-1}, \theta_{t-1} - \theta_{T+1}), (\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1), (\theta_{t-1} + 3^{T+2}, \theta_{t-1} - \theta_{T+1})\}$, and it can be checked that rank $M = 5$. Since all the hyperplanes through a fixed $(\theta_{t-2}, \theta_{t-2} - \theta_T)$-secundum of $\delta_{t+1}$ have diversity of type (A-6) with $s = 1$, we get the diversity of type (B-6) with $s = 1$ for $\delta_{t+1}$. Similarly we obtain the diversity of type (B-7) with $s = 1$ for $\delta_{t+1}$ from the assumption $c_{t+1}(\theta_{t-1}, \theta_{t-1} + \theta_{T+1} + 1) > 0$. Actually, one can get the diversity of type (B-6) (respectively type (B-7)) with $s$ for $\delta_{t+1}$ from a hyperplane having the diversity of type (A-6) (respectively type (A-7)) with $s$ for $1 \leqslant s \leqslant T$, and the diversity of type (B-4) (respectively type (B-5)) with $s + 1$ for $\delta_{t+1}$ from a hyperplane having the diversity of type (A-4) (respectively type (A-5)) with $s$ for $1 \leqslant s \leqslant T$.

If we assume that $c_{t+1}(\theta_{t-1}, 3^{t-1}) > 0$ and that none of hyperplanes having the diversity of type (A-1), (A-2), (A-4)–(A-7) exists, then we get the diversity of type (B-3) for $\delta_{t+1}$. Hence (B-1)–(B-7) are the only solutions of $(*)$ when $t$ is odd.

It can be also proved similarly as above that (A-1)–(A-7) are the only solutions of $(*)$ for $\delta_{t+1}$ containing a (4,3)-plane when $t$ is even ($\geqslant 6$). $\square$

**Proof of Theorem 1.3.** We prove for $k \geqslant 5$ (see [8] for $k = 3, 4$). We can take $t$ as $t = k - 1$. We denote $\varphi_0^{(t)} + \varphi_1^{(t)}$ of (A-$j$) and (B-$j$) by $\alpha(j)$ and $\beta(j)$, respectively. For $4 \leqslant j \leqslant 7$, we use $\alpha(j, s)$ and $\beta(j, s)$ for $\alpha(j)$ and $\beta(j)$ to specify $s$. Then it holds that

$$\begin{aligned}\alpha(3) &< \alpha(4, T) = \alpha(6, T) < \cdots < \alpha(4, 1) = \alpha(6, 1) < \alpha(1) < \alpha(2) < \alpha(5, 1)\\ &= \alpha(7, 1) < \cdots < \alpha(5, T) = \alpha(7, T) = 2\theta_{t-1} + \theta_{t-3} + 1\end{aligned}$$

and that

$$\begin{aligned}\beta(3) &< \beta(4, U + 1) = \beta(6, U) < \cdots < \beta(4, 1) = \beta(1) < \beta(2) = \beta(5, 1) < \cdots < \beta(7, U)\\ &= \beta(5, U + 1) = 2\theta_{t-1} + \theta_{t-3} + 1.\end{aligned}$$

Hence $\varphi_0^{(t)} + \varphi_1^{(t)}$ is at most $2\theta_{t-1} + \theta_{t-3} + 1$ for all $(\varphi_0^{(t)}, \varphi_1^{(t)}) \in \Lambda_t^+$, $t \geqslant 4$. This completes the proof of Theorem 1.3. $\square$

Let $\mathcal{C}$ be a $[n, 5, d]_3$ code with diversity (40,36) which is of type (B-1) with $t = 4$. Since $i + j \leqslant 2\theta_2 + 3 - \theta_0 = 28$ for any $(i, j)$ with $c_4(i, j) > 0$, $\mathcal{C}$ is not extendable if $\Phi_e < \theta_3 - 28 = 12$ by Lemma 2.3. Thus we get the following Theorems 2.9 and 2.10 from Theorem 2.8 since the minimum value of $\theta_{t-1} - (i + j)$ for $c_t(i, j) > 0$ is as follows:

$\theta_{t-2} - \theta_T$ for (A-1), (A-2); $3^{t-2}$ for (A-3), (B-3);

$\theta_{t-2} - \theta_{T+s}$ for (A-4), (A-5), (A-7);

$\theta_{t-2} - \theta_{T-1+s}$ for (A-6); $\theta_{t-2} - \theta_U$ for (B-1); $\theta_{t-2} - \theta_{U+1}$ for (B-2);

$\theta_{t-2} - \theta_{U+s}$ for (B-4), (B-5), (B-6); $\theta_{t-2} - \theta_{U+1+s}$ for (B-7).

**Theorem 2.9.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $\gcd(3,d)=1$, $k$ even ($\geqslant 6$), and put $T=k/2-2$. Then $\mathcal{C}$ is not extendable if*

(1) $\Phi_e<\theta_{k-3}-\theta_T$ *when* $(\Phi_0,\Phi_1)=(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$ *or* $(\theta_{k-2}+3^{T+1},\theta_{k-2}-\theta_T)$;
(2) $\Phi_e<\theta_{k-3}-\theta_{T+s}$ *when* $k\geqslant 8$ *and* $(\Phi_0,\Phi_1)\in\{(\theta_{k-2}-3^{T+1+s},\theta_{k-2}+\theta_{T+s}+1),(\theta_{k-2}+3^{T+1+s},\theta_{k-2}-\theta_{T+s}),(\theta_{k-2},\theta_{k-2}+\theta_{T+s}+1)\}$ *for some* $s$ $(1\leqslant s<T)$;
(3) $\Phi_e<\theta_{k-3}-\theta_{T+s-1}$ *when* $(\Phi_0,\Phi_1)=(\theta_{k-2},\theta_{k-2}-\theta_{T+s})$ *for some* $s$ $(1\leqslant s\leqslant T)$.

**Theorem 2.10.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $\gcd(3,d)=1$, $k$ odd ($\geqslant 5$), and put $U=(k-5)/2$. Then $\mathcal{C}$ is not extendable if*

(1) $\Phi_e<\theta_{k-3}-\theta_U$ *when* $(\Phi_0,\Phi_1)=(\theta_{k-2},\theta_{k-2}-\theta_{U+1})$;
(2) $\Phi_e<\theta_{k-3}-\theta_{U+1}$ *when* $(\Phi_0,\Phi_1)=(\theta_{k-2},\theta_{k-2}+\theta_{U+1}+1)$;
(3) $\Phi_e<\theta_{k-3}-\theta_{U+s}$ *when* $k\geqslant 7$ *and* $(\Phi_0,\Phi_1)\in\{(\theta_{k-2}-3^{U+1+s},\theta_{k-2}+\theta_{U+s}+1),(\theta_{k-2}+3^{U+1+s},\theta_{k-2}-\theta_{U+s}),(\theta_{k-2},\theta_{k-2}-\theta_{U+1+s})\}$ *for some* $s$ $(1\leqslant s\leqslant U)$;
(4) $\Phi_e<\theta_{k-3}-\theta_{U+1+s}$ *when* $k\geqslant 9$ *and* $(\Phi_0,\Phi_1)=(\theta_{k-2},\theta_{k-2}+\theta_{U+1+s}+1)$ *for some* $s$ $(1\leqslant s<U)$.

Finally, we give another kind of non-extendable results. For $1\leqslant u\leqslant 3$, we define $\xi_u$ as follows:

$$\xi_u=\min\bigl\{i \mid A_i>0,\ i>d,\ i\equiv d+u \pmod 3\bigr\}.$$

Obviously, for $1\leqslant u\leqslant 3$, we have

$$\xi_u\geqslant d+u. \tag{2.1}$$

**Lemma 2.11.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $d\equiv 1 \pmod 3$, $k\geqslant 4$, and let $c_{k-1}(i,j)$'s be the corresponding spectrum. Then $\mathcal{C}$ is not extendable if*

$$3^{k-2}n<i\cdot\xi_2+j\cdot\xi_1+(\theta_{k-2}-i-j)\xi_3 \quad \textit{for all } (i,j) \textit{ with } c_{k-1}(i,j)>0. \tag{2.2}$$

**Proof.** Suppose that $\mathcal{C}$ is extendable. Then there exists an $(i,j)$-hyperplane $\Pi$ in the dual space of $\Sigma=\mathrm{PG}(k-1,3)$ such that $c_{k-1}(i,j)>0$ and that $\Pi\setminus F\subset F_e$. Hence there exists a point $P$ in $\Sigma$ such that the hyperplanes containing $P$ consists of $i$ hyperplanes from $F_0$, $j$ hyperplanes from $F_1$ and $\theta_{k-2}-i-j$ hyperplanes from $F_e$. Since there are $\theta_{k-3}$ hyperplanes through a fixed line in $\Sigma$, we have

$$n\leqslant\bigl\{i(n-\xi_2)+j(n-\xi_1)+(\theta_{k-2}-i-j)(n-\xi_3)\bigr\}/\theta_{k-3},$$

which contradicts (2.2). □

Similarly, we obtain the following.

**Lemma 2.12.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $d\equiv 2 \pmod 3$, $k\geqslant 4$, and let $c_{k-1}(i,j)$'s are the corresponding spectrum. Then $\mathcal{C}$ is not extendable if*

$$3^{k-2}n < i\cdot\xi_1 + j\cdot\xi_2 + (\theta_{k-2}-i-j)\xi_3 \quad \text{for all } (i,j) \text{ with } c_{k-1}(i,j)>0. \tag{2.3}$$

Now, let $m_1$ and $m_2$ be as follows:

$$m_1=\max\{i+2j \mid c_{k-1}(i,j)>0\},$$
$$m_2=\max\{2i+j \mid c_{k-1}(i,j)>0\}.$$

Then we get the following from (2.1) and Lemmas 2.11, 2.12.

**Lemma 2.13.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $d\equiv \nu \pmod 3$, $\nu=1$ or $2$, $k\geqslant 4$, and let $c_{k-1}(i,j)$'s are the corresponding spectrum. Then $\mathcal{C}$ is not extendable if*

$$3^{k-2}n-\theta_{k-2}d<3\theta_{k-2}-m_\nu.$$

**Corollary 2.14.** *Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $\gcd(d,3)=1$. Then $\mathcal{C}$ is not extendable if*

(1) $9n-13d<26$ *when* $(\Phi_0,\Phi_1)=(13,9)$, $d\equiv 1 \pmod 3$;
(2) $9n-13d<25$ *when* $(\Phi_0,\Phi_1)=(10,15)$, $d\equiv 2 \pmod 3$;
(3) $9n-13d<23$ *when* $(\Phi_0,\Phi_1)=(10,15)$, $d\equiv 1 \pmod 3$ *or when* $(\Phi_0,\Phi_1)=(16,12)$, $d\equiv 1 \pmod 3$;
(4) $9n-13d<22$ *when* $(\Phi_0,\Phi_1)=(13,9)$, $d\equiv 2 \pmod 3$ *or when* $(\Phi_0,\Phi_1)=(16,12)$, $d\equiv 2 \pmod 3$.

**Example 2.1.**

(1) There are three $[8,4,4]_3$ codes with diversities $(10,15)$, $(16,12)$, $(4,18)$ up to equivalence [1]. Two of them with diversities $(10,15)$, $(16,12)$ are not extendable by Corollary 2.14. The other code with diversity $(4,18)$ is extendable by Theorem 1.1.
(2) There are three $[17,4,10]_3$ codes with diversity $(13,9)$ up to equivalence [1], all of which are not extendable by Corollary 2.14. See also Remark 2 in Section 3.

Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $\gcd(d,3)=1$, $k\geqslant 5$.

When $(\Phi_0,\Phi_1)=(\theta_{k-2},3^{k-2})$ (i.e., type (A-3) or (B-3)), it follows from Lemma 2.7 that

$$m_1=\theta_{k-2},\qquad m_2=2\cdot 3^{k-2}-1. \tag{2.4}$$

For other types of diversities, $m_1$ and $m_2$ are calculated from Theorem 2.8 as Tables 2.3 and 2.4, where $T$, $U$ and $s$ are taken as in the theorem. The following is a consequence of Lemma 2.13.

Table 2.3
$k$ even

| Type | (A-1) | (A-2) | (A-4) | (A-5) | (A-6) | (A-7) |
|---|---|---|---|---|---|---|
| $m_1$ | $\theta_{k-2}+3^{T+1}$ | $\theta_{k-2}+3^{T+1}$ | $\theta_{k-2}+3^{T+s+1}$ | $\theta_{k-2}+3^{T+s+1}$ | $\theta_{k-2}$ | $\theta_{k-2}+3^{T+s+1}$ |
| $m_2$ | $\theta_{k-2}+\theta_T$ | $\theta_{k-2}+\theta_{T+1}$ | $\theta_{k-2}+\theta_{T+s}$ | $\theta_{k-2}+\theta_{T+s+1}$ | $\theta_{k-2}+\theta_{T+s}$ | $\theta_{k-2}+\theta_{T+s}$ |

Table 2.4
$k$ odd

| Type | (B-1) | (B-2) | (B-4) | (B-5) | (B-6) | (B-7) |
|---|---|---|---|---|---|---|
| $m_1$ | $\theta_{k-2}$ | $\theta_{k-2}+3^{U+2}$ | $\theta_{k-2}+3^{U+s+1}$ | $\theta_{k-2}+3^{U+s+1}$ | $\theta_{k-2}$ | $\theta_{k-2}+3^{U+s+2}$ |
| $m_2$ | $\theta_{k-2}+\theta_{U+1}$ | $\theta_{k-2}+\theta_{U+1}$ | $\theta_{k-2}+\theta_{U+s}$ | $\theta_{k-2}+\theta_{U+s+1}$ | $\theta_{k-2}+\theta_{U+s+1}$ | $\theta_{k-2}+\theta_{U+s+1}$ |

**Theorem 2.15.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, $d\equiv\nu \pmod 3$, $\nu=1$ or 2, $k\geqslant 5$. Then $\mathcal{C}$ is not extendable if*

$$3^{k-2}n-\theta_{k-2}d<3\theta_{k-2}-m_\nu,$$

*where $m_\nu$ is given as* (2.4) *or as in Tables* 2.3 *and* 2.4 *according to $(\Phi_0,\Phi_1)$ in Theorem* 2.8 *with $t=k-1$.*

From (2.4) and Tables 2.3, 2.4, the minimum value of $3\theta_{k-2}-m_\nu$ is $3^{k-1}-3^{k-3}-1$ when $\nu=1$; and is $3^{k-1}-\theta_{k-3}-1$ when $\nu=2$. Hence we get the following.

**Corollary 2.16.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)$, $\gcd(d,3)=1$, $k\geqslant 5$, satisfying*

$$3^{k-2}n-\theta_{k-2}d<3^{k-1}-3^{k-3}-1 \quad \text{when } d\equiv 1 \pmod 3; \quad \text{or}$$
$$3^{k-2}n-\theta_{k-2}d<3^{k-1}-\theta_{k-3}-1 \quad \text{when } d\equiv 2 \pmod 3.$$

*Then $\mathcal{C}$ is extendable if and only if $(\Phi_0,\Phi_1)\in\mathcal{D}_k^*$.*

**Example 2.2.** Let $\mathcal{C}$ be an $[n,5,d]_3$ code with diversity $(\Phi_0,\Phi_1)$ with $(n,d)\in\{(54,35),$ $(79,52)$, $(80,53)$, $(103,68)$, $(106,70)$, $(107,71)$, $(112,74)$, $(115,76)$, $(116,77)$, $(119,79)$, $(120,80)\}$. Then $\mathcal{C}$ is not extendable if $(\Phi_0,\Phi_1)\in\mathcal{D}_5^+$ by Corollary 2.16.

## 3. Extendability of ternary linear codes of dimension 4

Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_4^+=\{(13,9),(10,15),(16,12)\}$, $d\equiv 1$ or $2 \pmod 3$. From the geometrical point of view (see Section 2), Theorems 1.5, 1.6, 1.7 are equivalent to the following Theorems 3.1, 3.2, 3.3, respectively:

**Theorem 3.1.** *Let $\mathcal{C}$ be an $[n,4,d]_3$ code with diversity* $(16,12)$, $\gcd(3,d)=1$. *Then $\mathcal{C}$ is extendable if and only if one of the following conditions holds*:

(1) *there are three collinear points* $Q_1, Q_2, Q_3 \in F_{\mathrm{e}}$;
(2) *there are three non-collinear points* $Q_1, Q_2, Q_3 \in F_{\mathrm{e}}$ *such that the three lines* $Q_1Q_2$, $Q_2Q_3$, $Q_3Q_1$ *are* $(0,2)$*-lines.*

**Proof.** ("only if" part) Assume $\mathcal{C}$ is extendable. Then there is an $(i,j)$-plane $\delta$ in $\Sigma^*$ satisfying $\delta \setminus F \subset F_{\mathrm{e}}$. Recall that $(i,j) \in \{(4,3),(4,6),(7,3)\}$ (see Table 2.2 in Section 2). If $\delta$ is a $(4,3)$-plane or a $(7,3)$-plane, then (1) holds since $c_2(1,0) > 0$ (see Table 2.1 in Section 2). If $\delta$ is a $(4,6)$-plane, then (2) holds by Lemma 2.4.

("if" part) Assume that (1) holds and let $l$ be the line containing $Q_1, Q_2, Q_3$. Then $l$ is a $(1,0)$-line, and there are exactly three $(4,3)$-planes and one $(7,3)$-plane, say $\delta_1$, through $l$. Since $\delta_1 \setminus F = \{Q_1, Q_2, Q_3\} \subset F_{\mathrm{e}}$, $\mathcal{C}$ is extendable by Lemma 2.3.

Next, assume (2) holds and let $\delta_2$ be the plane containing $Q_1, Q_2, Q_3$. Then $\delta_2$ must be a $(4,6)$-plane containing three points of $F_{\mathrm{e}}$. Hence $\mathcal{C}$ is extendable by Lemma 2.3. □

**Theorem 3.2.** *Let* $\mathcal{C}$ *be an* $[n,4,d]_3$ *code with diversity* $(13,9)$, $\gcd(3,d)=1$. *Then* $\mathcal{C}$ *is extendable if and only if one of the following conditions holds*:

(1) *there is a* $(1,0)$*-line* $l_1$ *which meets some* $(1,3)$*-line* $l_2$ *such that* $l_1 \setminus (l_1 \cap l_2) \subset F_{\mathrm{e}}$;
(2) *there are three non-collinear points* $P_1, P_2, P_3 \in F_1$ *such that the three lines* $P_1P_2$, $P_2P_3$, $P_3P_1$ *are* $(0,2)$*-lines each of which contains two points of* $F_{\mathrm{e}}$.

**Proof.** ("only if" part) Assume $\mathcal{C}$ is extendable. Then there is an $(i,j)$-plane $\delta$ in $\Sigma^*$ satisfying $\delta \setminus F \subset F_{\mathrm{e}}$. Recall that $(i,j) \in \{(4,0),(1,6),(4,3),(7,3)\}$ (see Table 2.2 in Section 2). If $\delta$ is a $(1,6)$-plane or a $(7,3)$-plane, then (1) holds by Lemma 2.4. If $\delta$ is a $(4,3)$-plane, then (2) holds by Lemma 2.4. Now, assume $\delta$ is a $(4,0)$-plane. Since there are six $(7,3)$-planes by Table 2.2, there is a $(7,3)$-plane, say $\delta'$, meeting $\delta$ in a $(1,0)$-line. Hence we can retake $\delta'$ as $\delta$ satisfying $\delta \setminus F \subset F_{\mathrm{e}}$ so that (1) holds.

("if" part) Assume that (1) holds and let $\delta$ be the plane containing $l_1$ and $l_2$. Then there are exactly one $(7,3)$-plane (say $\delta_1$), one $(1,6)$-plane and two $(4,0)$-planes through $l_1$, for a $(4,3)$-plane has no $(1,3)$-line. Since $\delta_1 \setminus F = l_1 \setminus (l_1 \cap l_2) \subset F_{\mathrm{e}}$, $\mathcal{C}$ is extendable by Lemma 2.3.

Next, assume (2) holds and let $\delta_2$ be the plane containing $P_1, P_2, P_3$. Then $\delta_2$ is a $(1,6)$-plane or a $(4,3)$-plane since $|\delta_2 \cap F_{\mathrm{e}}| \geqslant 6$ and $|\delta_2 \cap F_1| \geqslant 3$. But one cannot take such three $(0,2)$-lines as (2) in a $(1,6)$-plane from its geometrical structure (Lemma 2.4). So, $\delta_2$ is a $(4,3)$-plane containing six points of $F_{\mathrm{e}}$. Hence $\mathcal{C}$ is extendable by Lemma 2.3. □

**Theorem 3.3.** *Let* $\mathcal{C}$ *be an* $[n,4,d]_3$ *code with diversity* $(10,15)$, $\gcd(3,d)=1$. *Then* $\mathcal{C}$ *is extendable if and only if one of the following conditions holds*:

(1) *there are three non-collinear points* $Q_1, Q_2, Q_3 \in F_{\mathrm{e}}$ *such that the three lines* $Q_1Q_2$, $Q_2Q_3$, $Q_3Q_1$ *are* $(0,2)$*-lines*;
(2) *there are two* $(1,0)$*-lines* $l_1$ *and* $l_2$ *meeting in a point of* $F_0$, *say* $Q$, *such that* $l_1 \cap l_2 \setminus \{Q\} \subset F_{\mathrm{e}}$;
(3) *there are three non-collinear points* $P_1, P_2, P_3 \in F_1$ *such that the three lines* $P_1P_2$, $P_2P_3$, $P_3P_1$ *are* $(0,2)$*-lines each of which contains two points of* $F_{\mathrm{e}}$.

**Proof.** ("only if" part) Assume $\mathcal{C}$ is extendable. Then there is an $(i, j)$-plane $\delta$ in $\Sigma^*$ satisfying $\delta \setminus F \subset F_{\mathrm{e}}$ for some $(i, j) \in \{(1,6), (4,3), (4,6)\}$ (see Table 2.2 in Section 2). Hence (1) or (2) or (3) hold if $\delta$ is a $(4,6)$-plane or a $(1,6)$-plane or a $(4,3)$-plane respectively by Lemma 2.4.

("if" part) (1) and (3) are the same conditions with (2) of Theorem 3.1 and (2) of Theorem 3.2 respectively, so we omit the proof for these cases.

Assume that (2) holds and let $\delta$ be the plane containing $l_1$ and $l_2$. It follows from Lemma 2.4 that there is at most one $(1,0)$-line through a fixed point of $F_0$ in a $(4,3)$-plane. Since $c_2(1,0) = 0$ for a $(4,6)$-plane, $\delta$ must be a $(1,6)$-plane. Hence $\mathcal{C}$ is extendable by Lemma 2.3. □

**Proof of Theorem 1.8.** Assume that (2) of Theorem 1.5 is satisfied. As we have seen in the proof of Theorem 3.1, we can take the plane, say $\delta_2$, containing $Q_1$, $Q_2$, $Q_3$ so that the condition in Lemma 2.3 holds ($\delta_2 \subset \bar{F}$). Assume that $\delta_2$ is defined as the set of points in $\mathrm{PG}(3,3)$ whose coordinate vectors are $(x_0, x_1, x_2, x_3) \in V(4,3) \setminus \{\mathbf{0}\}$ such that $h_0x_0 + h_1x_1 + h_2x_2 + h_3x_3 = 0$. Then we can take $h = (h_0, h_1, h_2, h_3) \in V(4,3)$ so that $[G, h^{\mathrm{T}}]$ generates an extension of $\mathcal{C}$. Hence our assertion follows. Similarly we can prove for the other cases. □

**Remark 1.** Let $\mathcal{C}$ be an extendable $[n, 4, d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_4^+$, $d \equiv 1 \pmod 3$, and let $\mathcal{C}'$ be an extension of $\mathcal{C}$ with diversity $(\Phi_0', \Phi_1')$ obtained by adding a column vector $h$ ($h^{\mathrm{T}} \in V(4,3)$) to a generator matrix $G$ of $\mathcal{C}$, where $h$ corresponds to an $(i, j)$-plane $\delta$ in the dual space of $\mathrm{PG}(3,3)$ as in the proof of Theorem 1.8. Then it can be proved that $(\Phi_0', \Phi_1') = (16, 12)$ if $(i, j) = (7, 3)$ when $(\Phi_0, \Phi_1) = (16, 12)$ or if $(i, j) = (4, 3)$ when $(\Phi_0, \Phi_1) = (10, 15)$; $(\Phi_0', \Phi_1') = (13, 9)$ if $(i, j) = (4, 6)$ when $(\Phi_0, \Phi_1) = (10, 15)$ or if $(i, j) = (7, 3)$ when $(\Phi_0, \Phi_1) = (13, 9)$; $(\Phi_0', \Phi_1') = (10, 15)$ if $(i, j) = (4, 6)$ when $(\Phi_0, \Phi_1) = (16, 12)$ or if $(i, j) = (4, 3)$ when $(\Phi_0, \Phi_1) = (13, 9)$ or if $(i, j) = (1, 6)$ when $(\Phi_0, \Phi_1) = (10, 15)$.

**Example 3.1.** Let $\mathcal{C}_1$ be a $[13, 4, 7]_3$ code with a generator matrix

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \end{bmatrix},$$

whose weight distribution is $0^1 7^{20} 8^{14} 9^{22} 10^{16} 11^4 12^4$ (diversity (13,9), $\Phi_{\mathrm{e}} = 8$). Then we can take

$$P_1 = (0,1,1,0), \qquad P_2 = (1,0,2,2), \qquad P_3 = (1,2,1,2),$$
$$Q_1 = (0,0,1,2), \qquad Q_2 = (1,1,2,0), \qquad Q_3 = (1,1,1,1)$$

so that (1) of Theorem 1.6 holds. Hence, by Theorem 1.8, we can take $h^{\mathrm{T}} = (1,1,2,2)$ so that $[G_1, h]$ generates a $[14, 4, 8]_3$ code, whose weight distribution is $0^1 8^{26} 9^{22} 10^{14} 12^4 13^4$.

**Example 3.2.** Let $\mathcal{C}_2$ be a $[22, 4, 13]_3$ code with a generator matrix

$$G_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&2&2&2 \\ 0&0&1&0&1&1&0&0&1&1&0&0&1&1&0&0&1&2&2&1&1&2 \\ 0&0&0&1&1&2&1&2&0&1&1&2&0&1&0&1&1&0&1&0&1&2 \end{bmatrix},$$

whose weight distribution is $0^1 13^{20} 14^{10} 15^{26} 16^{16} 17^6 20^2$ (diversity (13,9), $\Phi_e = 8$). Then we can take

$$P_1 = (0,0,0,1), \qquad P_2 = (0,1,2,0), \qquad P_3 = (1,2,2,2)$$

so that (2) of Theorem 1.6 holds. Hence, by Theorem 1.8, we can take $h^T = (1,2,2,0)$ so that $[G_2, h]$ generates a $[23, 4, 14]_3$ code, whose weight distribution is

$$0^1 14^{26} 15^{12} 16^{30} 17^4 18^6 21^2.$$

**Example 3.3.** Let $\mathcal{C}_3$ be a $[23, 4, 14]_3$ code with a generator matrix

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&2&2&2&2&1 \\ 0&0&1&0&1&1&0&0&1&1&0&0&1&1&0&1&2&2&1&1&2&2&2 \\ 0&0&0&1&1&2&1&2&0&1&1&2&0&1&0&0&1&2&1&2&1&2&1 \end{bmatrix},$$

whose weight distribution is $0^1 14^{30} 15^{14} 16^{18} 17^6 18^{10} 21^2$ (diversity (13,9), $\Phi_e = 3$). Then we can take $F_e$ as $F_e = \{a_1 = (1,1,0,2), a_2 = (1,2,1,0), a_3 = (1,2,2,0)\}$. It suffices to check (1) of Theorem 1.6. Since $a_1$, $a_2$, $a_3$ are linearly independent, (1) does not hold. Hence $\mathcal{C}_3$ is not extendable.

**Example 3.4.** Let $\mathcal{C}_4$ be a $[13, 4, 7]_3$ code with a generator matrix

$$G_4 = \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&1&0&0&0&1&1&0&0&1&1&2&2 \\ 0&0&1&0&1&0&1&0&1&0&2&1&2 \\ 0&0&0&1&1&1&0&1&0&2&2&1&2 \end{bmatrix},$$

whose weight distribution is $0^1 7^{14} 8^{28} 9^{14} 10^{16} 11^2 12^6$ (diversity (10,15), $\Phi_e = 8$). Then we can take

$$Q_1 = (0,1,1,1), \qquad Q_2 = (1,0,1,2), \qquad Q_3 = (1,2,0,1),$$
$$Q_1' = (0,1,1,2), \qquad Q_2' = (1,0,1,1), \qquad Q_3' = (1,2,0,2)$$

so that (2) of Theorem 1.7 holds. Hence, by Theorem 1.8, we can take $h^T = (1,1,2,0)$ so that $[G_4, h]$ generates a $[14, 4, 8]_3$ code, whose weight distribution is

$$0^1 8^{26} 9^{18} 10^{24} 11^4 12^2 13^6.$$

Table 3.1
$[13, 4, 7]_3$ codes

| Diversity | (13, 0) | (4, 18) | (13, 9) | (10, 15) | (16, 12) | (13, 18) | (22, 9) |
|---|---|---|---|---|---|---|---|
| Extendable | 1 | 11 | 2 | 25 | 5 | 8 | 0 |
| Non-extendable | – | – | 10 | 12 | 3 | – | – |

Table 3.2
$[22, 4, 13]_3$ codes

| Diversity | (13, 0) | (4, 18) | (13, 9) | (10, 15) | (16, 12) | (13, 18) | (22, 9) |
|---|---|---|---|---|---|---|---|
| Extendable | 5 | 272 | 93 | 450 | 32 | 122 | 1 |
| Non-extendable | – | – | 236 | 390 | 64 | – | – |

Table 3.3
$[23, 4, 14]_3$ codes

| Diversity | (13, 0) | (4, 18) | (13, 9) | (10, 15) | (16, 12) | (13, 18) | (22, 9) |
|---|---|---|---|---|---|---|---|
| Extendable | 32 | 1 | 8 | 1 | 2 | 1 | 1 |
| Non-extendable | – | – | 65 | 6 | 6 | – | – |

**Remark 2.**

(1) There are 14 $[14, 4, 8]_3$ codes up to equivalence [1]. Five of them have diversity $(13, 0)$ which are extendable. One of them has diversity $(16, 12)$, two of them have diversity $(10, 15)$ and six of them have diversity $(13, 9)$, all of which are not extendable.
(2) There are 77 $[13, 4, 7]_3$ codes up to equivalence [5]. One of them has diversity $(13, 0)$, eight of them have diversity $(13, 18)$ and 11 of them have diversity $(4, 18)$, all of which are extendable by Theorem 1.1. Eight of them have diversity $(16, 12)$, 12 of them have diversity $(13, 9)$ and 37 of them have diversity $(10, 15)$. The extendability and non-extendability of $[13, 4, 7]_3$ codes are summarized as Table 3.1.
(3) There are 18 $[17, 4, 10]_3$ codes up to equivalence [1], none of them with diversity in $\mathcal{D}_4^+$ is extendable.
(4) There are 1665 $[22, 4, 13]_3$ codes and 123 $[23, 4, 14]_3$ codes up to equivalence [5]. See Tables 3.2 and 3.3 for their extendability. ("–" in Tables 3.1–3.3 means "zero by Theorem 1.1.")
(5) There are eight $[30, 4, 19]_3$ codes up to equivalence [1]. There are four such codes with diversity in $\mathcal{D}_4^+$, only one of which is extendable.

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